Published by Institute of Physics Publishing for SISSA/ISAS
Received: *November 13, 2002*
Accepted: *December 3, 2002*

Open and winding membranes, affine matrix theory and matrix string theory

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Abstract: We examine the structure of winding toroidal and open cylindrical membranes, especially in cases where they are stretched between boundaries. Non-zero winding or stretching means that there are linear terms in the mode expansion of the coordinates obeying Dirichlet boundary conditions. A linear term acts as an outer derivation on the subalgebra of volume-preserving diffeomorphisms generated by single-valued functions, and obstructs the truncation to matrix theory obtained via non-commutativity with rational parameter. As long as only one of the two membrane directions is stretched, the possible consistent truncation is to coordinates taking values in representations of an affine algebra. We show that this consistent truncation of the supermembrane gives a precise microscopic derivation of matrix string theory with the representation content appropriate for the physical situation. The matrix superstring theory describing parallel M5-branes is derived. We comment on the possible applications of the construction to membrane quantisation in certain M-theory backgrounds.

Keywords: M(atrix) Theories, Supergravity Models.

JHEP12(2002)005

Contents

1. Introduction

Supermembrane [1] theory [2] is a very promising candidate for a microscopic description of M-theory. Although it is not background invariant, it gives a completely new picture of the nature of space and time at small scales, together with a description of quantum-mechanical states that goes beyond local quantum field theory. These features are most clear in the matrix [3] truncation [2] of the membrane. It is widely appreciated that first-quantised supermembrane theory through its continuous spectrum [4] is capable of describing an entire ("multi-particle") Fock space. For reviews on the subject of membranes and matrices, see ref. [5]. Due to the immense technical difficulties associated with actual calculations in the theory, which is non-linear and inherently non-perturbative, few quantitative features are known in addition to the general picture, which is supported by many qualitative arguments. The maybe most important one is the proof that $\mathfrak{su}(N)$ matrix theory has a unique supersymmetric ground state [6, 7], which gives the relation to the massless degrees of freedom of $D = 11$ supergravity.

Many situations in M-theory backgrounds involve membranes that are not closed. Supermembranes may end on "defects", i.e., 5-branes and 9-branes [8]–[12]. It is urgent to have some mathematical formulation of these situations in order to understand the microscopic properties of physics in such backgrounds. One old enigma is the nature of the theory on multiple 5-branes, which we hope to be able to address using the present techniques. Matrix regularisation of winding and stretched membranes has been attempted before [13], but not entirely successfully. The matrix regularisation of open unstretched supermembranes has been considered in ref. [14]. The aim of this paper is to demonstrate why truncation to finite matrices fails, and what the consistent truncation instead is. It turns out to contain an affine algebra. A similar construction, based on discretisation, rather than non-commutativity, and thus more *ad hoc* postulated to match the structure of matrix string theory, has been considered in ref. [15] for a winding membrane. The present paper provides a natural microscopic explanation of that construction.

We start by a careful examination of the boundary conditions for open membranes. Due to the non-linearity of membrane dynamics, this is a non-trivial issue. In section 2, which can be skipped by any reader feeling confident with starting from imposed linear Dirichlet and Neumann conditions, we show how these are obtained using a partial gauge fixing. In section 3 we review the consistent truncation of membranes to matrices through introduction of a rational non-commutativity parameter, and generalise the procedure in presence of winding or stretching to truncation to elements in an affine algebra. Section 4 is devoted to the resulting matrix theory, which is shown to be equivalent to matrix string theory with a certain matter content, and in section 5 everything is extended to the super-symmetric membrane. We investigate in particular the half-supersymmetric case of open membranes ending on M5-branes. Finally, section 6 discusses some possible applications, open questions and directions of future investigations.

2. Analysis of boundary conditions

We start from the action for a bosonic membrane in flat D-dimensional Minkowski space with vanishing 3-form tensor field,

$$S = -T \int d^3\xi \sqrt{-g}\,, \tag{2.1}$$

where g is the the determinant of the induced metric $g_{IJ} = \partial_I X^a \partial_J X^b \eta_{ab}$. The momenta conjugate to X^a are

$$P_a = \frac{\delta S}{\delta \dot{X}^a} = -T\sqrt{-g}(g^{00}\dot{X}_a + g^{0i}\partial_i X_a)\,, \tag{2.2}$$

where $i, j, \ldots = 1, 2$ are spacelike indices for the membrane and g with upper indices denotes the inverse of g_{IJ}. The constraints resulting are the ones corresponding to space reparametrisations,

$$\mathcal{T}_i \equiv P_a \partial_i X^a \approx 0\,, \tag{2.3}$$

and the hamiltonian constraint

$$\mathcal{H} \equiv \frac{1}{2T} P^2 + \frac{T}{2}\tilde{g} \approx 0\,, \tag{2.4}$$

$\tilde{g}_{ij} = \partial_i X^a \partial_j X^b \eta_{ab}$ being the induced metric on the membrane space-sheet, and $\tilde{g}$ its determinant. It is conveniently rewritten in terms of the "Poisson bracket" on the space-sheet,[1]

$$\{A, B\} = \varepsilon^{ij}\partial_i A \partial_j B\,, \tag{2.5}$$

[1] In this paper we will specialise to toroidal and cylindrical membrane topology — in general cases the Poisson bracket should include a scalar density in order to map scalar functions to scalar functions.

as $\tilde{g} = \frac{1}{2}\{X_a, X_b\}\{X^a, X^b\}$, so that the hamiltonian constraint takes the form

$$\mathcal{H} = \frac{1}{2T}P^2 + \frac{T}{2}\{X_a, X_b\}\{X^a, X^b\} \approx 0\,. \tag{2.6}$$

Consider an open membrane whose end-strings are confined to some (flat) hypersurfaces. We will soon choose the topology of a finite cylinder for the membrane, but for the moment the analysis applies locally at each boundary. For simplicity, we restrict our attention to parallel stationary hypersurfaces of space dimension p, and choose the coordinates of the embedding space so that the hypersurfaces are coordinate surfaces specified by $X^{a''} = X^{a''}_{(i)}$, $a'' = p+1, \ldots, D-1$ for the boundary component (i). We also choose local coordinates $\xi^i_{(i)} = (\sigma_{(i)}, \varrho_{(i)})$ on the membrane space-sheet so that boundary (i) is $\varrho_{(i)} = 0$. Dropping the index i and looking at one boundary component at the time, we thus have the $D-p-1$ Dirichlet boundary conditions

$$X^{a''}(\sigma, 0) = x^{a''}\,, \qquad a'' = p+1, \ldots, D-1\,. \tag{2.7}$$

The boundary conditions for the remaining components may be obtained as momentum conservation in the longitudinal directions. Demanding that $p_{a'} = \int d^2\xi P_{a'}$, $a' = 0, \ldots, p$, is conserved, i.e., that $\{H, p_{a'}\}_{PB} = 0$, where $H = \int d^2\xi \mathcal{H}$, so that there is no flow of momentum across the boundary, gives the Neumann-like conditions

$$\partial_1 X^{b'}\{X_{b'}, X_{a'}\}(\sigma,0) = \Big[(\partial_1 X)^2 \partial_2 X_{a'} - (\partial_1 X \cdot \partial_2 X)\partial_1 X_{a'}\Big](\sigma, 0) = 0\,, \qquad a' = 0, \ldots, p\,. \tag{2.8}$$

The boundary conditions (2.7) and (2.8) are of course to be treated as constraints, and as such, one must assure that their time evolution is consistent. Taking the Poisson brackets with H will yield new, secondary, constraints, and so on. This is a non-trivial procedure, since the membrane dynamics is non-linear. First, we note that the non-linear boundary conditions on $X^{a'}$ state that the part of $\partial_2 X^{a'}$ orthogonal to $\partial_1 X^{a'}$ vanishes. Such conditions become in practice intractable, and one should look for linear conditions that can be solved explicitly. If one is allowed to make a gauge choice $(\partial_1 X \cdot \partial_2 X)(\sigma, 0) = 0$, it combines with eq. (2.8) to $\partial_2 X^{a'}(\sigma, 0) = 0$, a linear Neumann condition. This is in fact a valid gauge choice corresponding to the generator $(\partial_2 \mathcal{T}_1)(\sigma, 0)$. This can be seen by examining which diffeomorphisms preserve the boundary conditions. The coordinates transform under a diffeomorphism with parameter f^i as $\delta_f X = f^i \partial_i X$. The presence of coordinates with Dirichlet boundary conditions implies that $f^2(\sigma, 0) = 0$. Expansion to linear order in ϱ: $f^1 = f^1_{(0)}(\sigma) + \varrho f^1_{(1)}(\sigma) + \mathcal{O}(\varrho^2)$, $f^2 = \varrho f^2_{(1)}(\sigma) + \mathcal{O}(\varrho^2)$ gives the transformation $\delta_f \partial_2 X^{a'}(\sigma, 0) = f^1_{(1)}(\sigma)\partial_1 X^{a'}(\sigma, 0)$, showing that exactly this transformation may be "used up" to complete the Neumann boundary conditions. This procedure is carried out step by step, so that in the end all odd derivatives of $\mathcal{T}_1$ are gauged fixed by $(\partial_1 X \cdot \partial_2^{2n+1} X)(\sigma, 0) = 0$. It is also necessary for the non-linearities not to interfere with the time development of the Dirichlet boundary conditions. The resulting system has pure linear Neumann conditions for $X^{a'}$, $P^{a'}$ (all odd derivatives with respect to ϱ vanish) and Dirichlet conditions for $X^{a''}$, $P^{a''}$ (even derivatives vanish). The remaining space diffeomorphisms consist of even f^1 and odd f^2.

Specialising to a membrane with the topology of a finite cylinder (some comments on topology will be made in the last section), $0 \leq \varrho \leq \pi$, and Dirichlet conditions at the two boundaries $X^{a''}(\sigma, 0) = 0$, $X^{a''}(\sigma, \pi) = L^{a''}/2$, the coordinates can be expanded in modes:

$$\begin{aligned} X^{a'}(\sigma,\varrho) &= x^{a'} + \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} x_n^{a'} e^{in\sigma} \cos m\varrho \,, \\ P^{a'}(\sigma,\varrho) &= \frac{1}{2\pi^2} p^{a'} + \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} p_n^{a'} e^{in\sigma} \cos m\varrho \,, \end{aligned} \tag{2.9}$$

$$\begin{aligned} X^{a''}(\sigma,\varrho) &= \frac{L^{a''}}{2\pi} \varrho + \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} x_n^{a''} e^{in\sigma} \sin m\varrho \,, \\ P^{a''}(\sigma,\varrho) &= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} p_n^{a''} e^{in\sigma} \sin m\varrho \,. \end{aligned} \tag{2.10}$$

There is still a large gauge invariance, the algebra of diffeomorphisms preserving this class of functions.

For future reference, we also give the conventions for the mode expansion of a winding membrane,

$$\begin{aligned} X^{a}(\sigma,\varrho) &= \frac{L^{a}}{2\pi} \varrho + \frac{1}{2\pi} \sum_{m,n=-\infty}^{\infty} x_n^{a} e^{in\sigma + im\varrho} \,, \\ P^{a}(\sigma,\varrho) &= \frac{1}{2\pi} \sum_{m,n=-\infty}^{\infty} p_n^{a} e^{in\sigma + im\varrho} \,. \end{aligned} \tag{2.11}$$

3. Consistent truncation and affine algebras

When considering the classical or quantum dynamics of an open membrane, one may aim at different kinds of descriptions. Two directions that immediately come into mind are:

(i) Trying to get rid of the coordinates obeying Dirichlet conditions and in that way obtain something similar to a kind of string theory (although not conformally invariant). If there is only one Dirichlet coordinate, as in the Hořava-Witten construction [8], the gauge symmetry suffices to eliminate it completely.

(ii) Working with the kinds of structures that are inherent to membrane theory, i.e., choose light-cone gauge and try to truncate the residual gauge symmetry, the algebra of area-preserving diffeomorphisms [16], and the coordinates to some kind of matrix theory.

The aim of this section is to demonstrate how a consistent truncation to an affine algebra manages to combine these approaches.

Light-cone gauge amounts to choosing $P^+ = \frac{1}{4\pi^2} p^+$ (closed toroidal membrane) or $P^+ = \frac{1}{2\pi^2} p^+$ (open cylindrical membrane) for the hamiltonian constraint and $\partial_i X^+ = 0$ for $\mathcal{T}_i$. Since the gauge choice for the space part is a total derivative, there is the remaining gauge symmetry $\varepsilon^{ij} \partial_i \mathcal{T}_j = \{P_I, X^I\} \approx 0$ (here, $a = (+, -, I)$), generating area-preserving diffeomorphisms.

The algebra of area-preserving diffeomorphisms is generated by functions $f(\xi)$ on the space-sheet with the bracket (2.5), together with global vector fields with holonomies around A- and B-cycles with the obvious action on each other and on the functions. Consider first the algebra of function under the Poisson bracket. It is a well known fact that this algebra, in a certain sense, is $\mathfrak{su}(\infty)$ [2, 17]. The reason is clear from the truncation below. To get to matrix theory, with a finite-dimensional gauge algebra $\mathfrak{su}(N)$, there is a well-defined mathematical procedure. One introduces a non-commutativity $[\sigma,\varrho]=i\theta$ on the membrane encoded in the (Weyl-ordered) star product

$$f\star g=fe^{\frac{i}{2}\theta\varepsilon^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j}g\,. \tag{3.1}$$

The Fourier modes of definite momentum then commute as

$$[e^{ik_i\xi^i},e^{ik'_i\xi^i}]_\star=-2i\sin\left(\frac{1}{2}\theta k\times k'\right)e^{i(k_i+k'_i)\xi^i}\,, \tag{3.2}$$

where $k\times k'=\varepsilon^{ij}k_ik'_j$. The space-sheet Poisson bracket is recovered as the linear term in the limit $\theta\to0$,

$$\{f,g\}=-i\lim_{\theta\to0}\theta^{-1}[f,g]_\star\,. \tag{3.3}$$

If θ is "rational", $\theta=2\pi q/N$ where q and N are coprime integers, the sine function in the structure constants will have zeroes. The functions $e^{isN\sigma}$ and $e^{irN\varrho}$ ($s,r\in\mathbb{Z}$) commute with all other functions, they are central elements in the algebra. This means that they can be consistently modded out from the algebra of function under the star product, since left and right multiplication coincide on all functions. If the star products by $e^{iN\sigma}$ and $e^{iN\varrho}$ are identified with the identity operator, one obtains the equivalences

$$\begin{aligned}e^{i(k+N)\sigma+il\varrho}&\approx(-1)^le^{ik\sigma+il\varrho}\,,\\ e^{ik\sigma+i(l+N)\varrho}&\approx(-1)^ke^{ik\sigma+il\varrho}\,.\end{aligned} \tag{3.4}$$

The star-commutator algebra after this "consistent truncation" is $\mathfrak{su}(N)$. We will restrict to odd values $N=2M+1$, when the square of momenta can be placed symmetrically around the origin, $|k|,|l|\leq M$, and to $q=1$ (all q's not dividing N are equivalent). Effectively, non-commutativity produces finite translations from multiplication with exponentials, and if a finite number (N) of these add up to an integer number of periods there is something very similar to a discretisation of the circle. Of course, if a coordinate is both discrete and periodic, so is the momentum.

If one restricts to functions that vanish at $\varrho=0,\pi$ (sine functions) occurring in the expansion of a coordinate with Dirichlet boundary conditions, the algebra is an orthogonal algebra $\mathfrak{so}(2M+1)=B_M$ [18].

Let us now come to the point of this section. In a membrane theory, the functions we manipulate and truncate are the actual coordinates. If some of them have an expansion as in eqs. (2.10) and (2.11) with non-zero L's, i.e., if the membrane is winding or stretched between two hypersurfaces (branes) that are not on top of each other, the linear function representing the winding or stretching does not extend to a single-valued function on the circle $\varrho \in [0, 2\pi]$. We will work for a little while with winding membranes instead of stretched, just because the exponentials are easier to deal with, and then turn back to the sine and cosine functions in the expansion in eqs. (2.9), (2.10). The stretching/winding introduces a new element $\rho/2\pi$ in the algebra. Its star-adjoint action on functions is identical to the action of the vector field $-i\frac{\theta}{2\pi}\partial_1$, namely, $[\frac{\varrho}{2\pi}, e^{ik\sigma}]_\star = \frac{k\theta}{2\pi}e^{ik\sigma} = \frac{k}{N}e^{ik\sigma}$. We conclude that it is no longer consistent to identify functions modulo $e^{iN\sigma}$, since the functions we previously considered equivalent have different eigenvalues for $\rho/2\pi$.[2] The only central elements remaining are $e^{irN\varrho}$. Even if the functions $e^{isN\sigma}$ are no longer central, they form an abelian ideal of the star commutator algebra, and one can restrict to the quotient by this ideal, which amounts to simply discarding these terms in the mode expansion. We have an effective "discretisation" in the ϱ direction, while σ remains "continuous".

The linear function $\rho/2\pi$ acts as an outer derivation on the periodic functions. This follows directly from the fact that it acts with the same star commutator and that it is not periodic, although the result is. As an outer derivation, it can be used to construct a non-trivial central extension:

$$[f, g]' = [f, g]_\star + \gamma\,\mathrm{tr}\left(f\left[\frac{\varrho}{2\pi}, g\right]_\star\right). \tag{3.5}$$

Clearly, in any consistent truncation containing $\rho/2\pi$ it will remain an outer derivation. Since all derivations of finite-dimensional Lie algebras are inner, and they have no extensions of non-trivial cohomology, this shows that one can not hope to reach a matrix theory with finite-dimensional matrices.

The algebra resulting after the identification

$$e^{ik\sigma+i(l+N)\varrho} \approx (-1)^k e^{ik\sigma+il\varrho} \tag{3.6}$$

and exclusion of the functions $e^{isN\sigma}$ is infinite-dimensional. It has finite rank $N = 2M+1$, one set of Cartan generators can be taken as $e^{\pm ik\varrho}$, $k = 1, \ldots, M$, and $\rho/2\pi$. This algebra is an affine $\mathfrak{su}(N)$, or $A^{(1)}_{N-1}$. What is a little confusing about the commutators (3.2) is that it is hard to see how the $\mathfrak{su}(N)$ subalgebra is embedded. To find this embedding is essentially the same problem as to construct a relation between $\rho/2\pi$ and the standard derivation of the affine algebra. Modulo normalisation, they can only differ by an inner derivation.

The structure becomes clear in an example. Let us consider $N = 3$, with Cartan generators chosen to $e^{\pm i\varrho}$ and $\rho/2\pi$. In order to find the root system, we diagonalise $h = e^{i\varrho}$. The real components of this generator are orthogonal, so the result will be immediately possible to visualise in a euclidean plane. h acts on the three functions $\{e^{in\sigma+im\varrho}\}_{m=-1,0,1}$ for fixed n as the matrix

$$h^{(n)} = 2i\sin\frac{n\pi}{3}\begin{bmatrix} 0 & 0 & (-1)^n \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{3.7}$$

The matrix in brackets is easily diagonalisable, and gives the eigenvalues $\{\mu\} = \{-1, e^{\pm\frac{\pi i}{3}}\}$ for n odd and minus the same numbers for n even. When this is multiplied by the prefactor

[2] This fact was noted in ref. [13].

(which in this case is a sign factor up to normalisation), the result is that up to normalisation of h the eigenvalues are zero for $n = 0 \bmod 3$, $\{\mu\}$ for $n = 1 \bmod 3$ and $\{-\mu\}$ for $n = 2 \bmod 3$. With suitable normalisation of h and the derivation, whose eigenvalues are the n mode numbers, we get the root space of figure 1. As simple roots we can take α_1, α_2 and α_3. The vertical direction is a null direction, and the Cartan matrix of the algebra is

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \tag{3.8}$$

which is the Cartan matrix of affine $\mathfrak{su}(3)$, or $A_2^{(1)}$. The eigenvalue n of the derivation is not the "standard" one used for affine algebras, so the embedding of the $\mathfrak{su}(3)$ subalgebra is "tilted", as shown in the figure (the whole picture may also be rotated an angle $\pm 2\pi/3$ around the vertical). The general picture is that the N simple roots of $A_{N-1}^{(1)}$ are found at eigenvalue $1/N$ of $\rho/2\pi$, and are oriented as the $N-1$ simple roots of $\mathfrak{su}(N)$ and minus the highest root. Since the simple $\mathfrak{su}(N)$ roots (α_1 and α_2 for the $\mathfrak{su}(3)$ case) have eigenvalue $1/N$ under the derivation $\rho/2\pi$, a root at height ℓ (a sum of ℓ simple roots) has eigenvalue ℓ/N. The relation between the derivation used so far and the standard derivation of an affine algebra is then

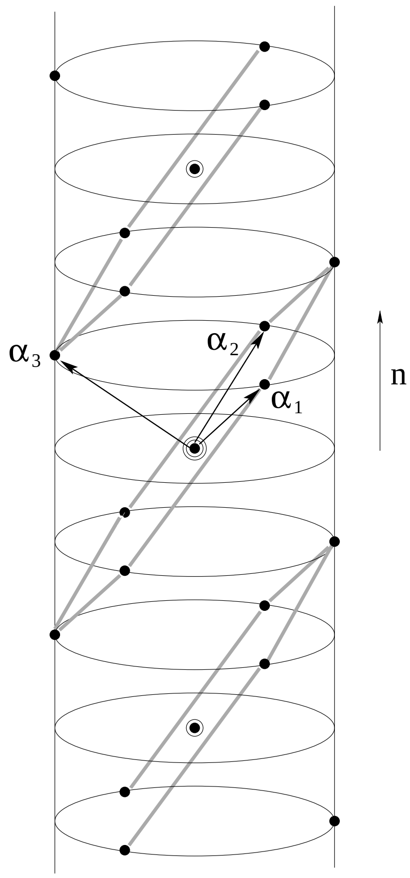

Figure 1: The root system of affine $\mathfrak{su}(3)$.

$$d = \frac{\varrho}{2\pi} - \frac{1}{N} H_\rho , \tag{3.9}$$

where H_ρ is the element in the $\mathfrak{su}(N)$ Cartan subalgebra whose eigenvalues are the heights. This is achieved for the Weyl vector $\rho = \sum_{i=1}^{N-1} \lambda_{(i)}$, the sum of all fundamental weights of $\mathfrak{su}(N)$. In general (see e.g. ref. [19]), the the derivation for an affine algebra based on a finite-dimensional Lie algebra $\mathfrak{g}$ introduces a gradation of $\mathfrak{g}$ into eigenspaces under some automorphism. If this automorphism is inner, i.e., generated by an (inner) derivation, the algebra is isomorphic, via a redefinition of the derivation as in eq. (3.9), to an untwisted affine algebra. Twisted affine algebras are classified by outer automorphisms of $\mathfrak{g}$.

So far, we have strictly speaking been talking about winding closed membranes, not stretched open membranes. The boundary conditions, as shown in section 2, enforce the transverse, Dirichlet, coordinates $X^{a''}$ to contain the stretchings L and sine functions, while the longitudinal, Neumann, ones $X^{a'}$ are expanded in cosines (the light-cone $+-$ directions eliminated in the passage to light-cone gauge are of course in the longitudinal sector). Exactly as the split into sines and cosines breaks $\mathfrak{su}(N)$ to $\mathfrak{so}(N)$, where the sines span the adjoint representation and the cosines the symmetric traceless representation, the same happens for the affine algebra (the tracelessness corresponds to the removal of the ideal $e^{isN\sigma}$).

4. Affine matrix theory for winding or stretched membranes

Let us forget the translational modes for a while, they commute with all elements of the algebra, and any (localised) quantum mechanical state we get will in the end be promoted to a field (in the light-cone gauge) depending on $x^{I'}$. After the consistent truncation has been performed, we are in a position to write down a mode expansion for the coordinates and momenta of a winding or stretched membrane. In the winding case, the gauge algebra is $A^{(1)}_{N-1}$ with generators $\{T^{\mathcal{A}}_n, d\}$, where $\mathcal{A}$ is an adjoint $\mathfrak{su}(N)$ index and $n \in \mathbb{Z}$ are the eigenvalues of d. We take the generators to be normalised as $\mathrm{Tr}(T^{\mathcal{A}}_m T^{\mathcal{B}}_n) = \mathrm{tr}(T^{\mathcal{A}}T^{\mathcal{B}})\delta_{m+n,0} = \delta^{\mathcal{AB}}\delta_{m+n,0}$, corresponding to the orthonormality of the trigonometric functions. The coordinates and momenta, truncated from eq. (2.11), are

$$
\begin{aligned}
X^I &= L^I d + \sum_{n=-\infty}^{\infty} x^{I\mathcal{A}}_n T^{\mathcal{A}}_n\,,\\
P^I &= \sum_{n=-\infty}^{\infty} p^{I\mathcal{A}}_n T^{\mathcal{A}}_n\,,
\end{aligned}
\tag{4.1}
$$

with Poisson brackets $\{x^{I\mathcal{A}}_m, p^{J\mathcal{B}}_n\}_{PB} = \delta^{IJ}\delta^{\mathcal{AB}}\delta_{m+n,0}$. Equivalently, we can think of n as a mode number of the expansion in a coordinate σ,[3]

$$
\begin{aligned}
X^I(\sigma) &= -iL^I\frac{\partial}{\partial\sigma} + \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} x^{I\mathcal{A}}_n T^{\mathcal{A}} e^{in\sigma}\,,\\
P^I(\sigma) &= \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} p^{I\mathcal{A}}_n T^{\mathcal{A}} e^{in\sigma}\,,
\end{aligned}
\tag{4.2}
$$

Here, we have treated winding and non-winding coordinates together. In the case of a stretched membrane, as mentioned, the Dirichlet coordinates go into the adjoint representation of the affine algebra $B^{(1)}_M$ and the Neumann ones into the symmetric traceless representation. We thus have the expansion

$$
\begin{aligned}
X^{I'}(\sigma) &= \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} x^{I'AB}_n S^{AB} e^{in\sigma}\,,\\
P^{I'}(\sigma) &= \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} p^{I'AB}_n S^{AB} e^{in\sigma}\,,
\end{aligned}
\tag{4.3}
$$

[3]Notice that this σ is *not* the same σ as the original membrane coordinate, since we performed a "de-twisting" of the algebra in eq. (3.9). While the spectrum of $\rho/2\pi$ is fractional with spacing $1/N$, d has integer spectrum. In this way, the new σ describes a circle with N times smaller circumference than the old σ, and, depending on the height ℓ, different $\mathfrak{su}(N)$ generators are shifted by different fractions ℓ/N to become integer modes in the new σ.

$$X^{I''}(\sigma) = -iL^{I''}\frac{\partial}{\partial\sigma} + \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} x_n^{I''AB}T^{AB}e^{in\sigma}\,,$$
$$P^{I''}(\sigma) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} p_n^{I''AB}T^{AB}e^{in\sigma}\,, \tag{4.4}$$

where A, B are vector indices of $\mathfrak{so}(N)$, T^{AB} is a normalised basis for the adjoint representation and S^{AB} for the traceless symmetric representation.

According to the correspondence between the space-sheet Poisson bracket and the star commutator (eq. (3.3) without the limit), the latter now being an ordinary commutator, $\{f,g\} \to -\frac{iN}{2\pi}[f,g]$. Since the constant modes have been left out of the discussion, the light-cone hamiltonian gives the mass operator for a winding membrane:

$$\frac{1}{2}M^2 = (2\pi)^2\,\mathrm{Tr}\left\{\frac{1}{2}P^IP^I + \frac{1}{4}T^2\frac{N^2}{(2\pi)^2}[X^I,X^J][X^I,X^J]\right\}, \tag{4.5}$$

where the trace is over the affine algebra. Choose a basis where one of the coordinates, Y, has a winding L and the others, X^R, are periodic. Rescaling to dimensionless variables, $X^I \to LX^I$, $P^I \to L^{-1}P^I$, identifies Y with $-i$ times a covariant σ-derivative, $Y = -iD = -i\partial_\sigma + A$. We denote the conjugate momentum E. The affine trace is expressed as $\mathrm{Tr} = \int d\sigma\,\mathrm{tr}$, so that

$$\begin{aligned}\frac{1}{2}M^2 = \frac{(2\pi)^2}{L^2}\int d\sigma\,\mathrm{tr}\Big\{&\frac{1}{2}(E^2 + P^RP^R) + \\ &+\frac{1}{2}T^2L^6\frac{N^2}{(2\pi)^2}\left(-DX^RDX^R + \frac{1}{2}[X^R,X^S][X^R,X^S]\right)\Big\}.\end{aligned} \tag{4.6}$$

If we define the dimensionless coupling constant $g = \sqrt{N/2\pi TL^3}$,

$$\begin{aligned}\frac{1}{2}\left(\frac{gLM}{2\pi}\right)^2 = \int d\sigma\,\mathrm{tr}\Big\{&\frac{1}{2}g^2(E^2 + P^RP^R) + \\ &+\frac{1}{2}g^{-2}\left(-DX^RDX^R + \frac{1}{2}[X^R,X^S][X^R,X^S]\right)\Big\}.\end{aligned} \tag{4.7}$$

This is the hamiltonian of $d = 2$ Yang-Mills with coupling constant g coupled to $D-3$ adjoint scalars with a quartic potential in the gauge $A_0 = 0$, i.e., of $d = D-1$ Yang-Mills dimensionally reduced to $d = 2$. The consistent truncation performed has thus led to matrix string theory [20].

When the membrane is stretched, the truncated algebra is affine $\mathfrak{so}(N)$. The Dirichlet coordinates transform in the adjoint representation and the Neumann ones in the traceless symmetric representation. We get the expression for the mass operator:

$$\begin{aligned}\left(\frac{gLM}{2\pi}\right)^2 = \int d\sigma\,\mathrm{tr}\Big\{&\frac{1}{2}g^2(E^2 + P^RP^R + P^{I'}P^{I'}) + \\ &+\frac{1}{2}g^{-2}\Big(-DX^RDX^R - DX^{I'}DX^{I'} + \frac{1}{2}[X^{I'},X^{J'}][X^{I'},X^{J'}] + \\ &+\frac{1}{2}[X^R,X^S][X^R,X^S] + [X^R,X^{I'}][X^R,X^{I'}]\Big)\Big\}.\end{aligned} \tag{4.8}$$

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This is the hamiltonian of $d = 2$ Yang-Mills with gauge group $\mathfrak{so}(N)$ coupled to $D - p - 2$ adjoint scalars and $p - 1$ scalars in the symmetric traceless representation i.e., the dimensional reduction to $d = 2$ of $d = D - p$ Yang-Mills coupled to to $p - 1$ symmetric traceless scalars.

5. Supermembranes

In order to treat supersymmetric membranes, the boundary conditions for the fermions have to be analysed. This has been done in detail in ref. [14]. There it was shown that, under the assumption that some supersymmetry remains unbroken, the possible values of p are 1, 5 and 9. It is not clear whether $p = 1$ has some natural interpretation in M-theory; the other two cases correspond to membranes ending on M5-branes or on end-of-the-world 9-branes.

The case of 9-branes is slightly complicated. It is straightforward to verify that when the light-cone rotation algebra $\mathfrak{so}(9)$ is broken by the stretching to $\mathfrak{so}(8)$, the spinor $16 \to 8_s \oplus 8_c$, of which one spinor is odd over the boundary and the other one even, so they go into the adjoint and the symmetric traceless representations of the affine algebra. It was argued in ref. [18] that this boson-fermion mismatch should be compensated by a twisted sector in the fundamental representation. It is also known that the classical anomaly cancellation of the supermembrane demands degrees of freedom residing at the boundary, corresponding to an E_8 WZW model [9, 10]. We hope to be able to come back to this point, and turn to supermembranes ending on M5-branes.

The light-cone rotation group of the supermembrane, $\mathfrak{so}(9)$, is then broken into $\mathfrak{so}(4)_{||} \oplus \mathfrak{so}(4)_\perp \cong \mathfrak{su}(2)^4$, one of the $\mathfrak{so}(4)$'s being the little group of the longitudinal light-cone and the other one rotations between the non-stretched transverse directions. The light-cone spinor decomposes as $16 \to (1, 2; 1, 2) \oplus (1, 2; 2, 1) \oplus (2, 1; 1, 2) \oplus (2, 1; 2, 1)$, The boundary conditions, as derived in ref. [14], force these spinors to be odd, odd, even and even, respectively, over the boundary. The odd ones combines with the gauge field (the stretching coordinate) and the the $D - p - 2 = 4$ odd longitudinal scalars in $(2, 2, 1, 1)$ into one supermultiplet and the even ones with the $p - 1 = 4$ even transverse unstretched scalars in $(1, 1, 2, 2)$ into another multiplet. The global $\mathfrak{su}(2)^4$ symmetry plays the rôle of an R-symmetry of the model. The mass operator is such that $(gLM/2\pi)^2$ is the hamiltonian of $\mathcal{N} = (1, 0)$, $d = 6$ super-Yang-Mills theory with gauge group $\mathfrak{so}(N)$, coupled to a hypermultiplet in the symmetric traceless representation, dimensionally reduced to $d = 2$, with coupling constant g (recall that adding hypermultiplets to $d = 6$ super-Yang-Mills does not introduce an additional coupling constant).

6. Discussion

We have shown how, in the presence of winding or stretching in one direction, a membrane can be truncated to an affine matrix theory, a matrix string theory. There are obvious applications in any situation where winding or stretched membranes occur. M-theory compactified on S^1, type IIA string theory, is one such example that has already been investigated [20]. Other M-theory backgrounds that should be examined are the one in the Hořava-Witten construction and backgrounds with parallel M5-branes.

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The modern interpretation of supermembrane theory implies that the topology of the membrane is irrelevant. Classically, a membrane can grow "spikes" at infinitesimally low energy cost. In the matrix truncation this is reflected in the existence of flat directions in the quartic potential. This flatness is not lifted by quantum effects in the supersymmetric membrane. Already a first-quantised supermembrane contains "multi-membrane" states. The properties are nicely reflected in the truncation to matrices of finite size, where all topological information about the membrane is lost, which should presumably be seen as physically correct. One may ask the question whether or not this is true for membranes with boundaries. The question actually disintegrates in two: does stretching stabilise the bulk of the membrane against topology change, and is the topology of the boundary fixed?

In ref. [13] it was argued that a winding membrane still has continuous spectrum, although the lowest energy is raised due to the winding. The same should hold for a stretched membrane. There are clearly flat directions in the potential, corresponding to commuting elements in the affine algebra. A ground state at the bottom of the spectrum should be BPS-saturated (it would be very interesting if the Witten index can be calculated). Then, on the other hand, it seems clear that the membrane is actually locked to the boundary branes, since it otherwise could have disintegrated into some closed membrane states. It is tempting to conjecture that the continuity of the spectrum stems from a description of "multi-string" states, as seen from the $p+1$-dimensional world-volume, and not from interaction with bulk degrees of freedom.

It is possible that the model described in section 5 can be used to describe $\mathcal{N}=(2,0)$ theories on systems of parallel M5-branes. The situation that has been considered here corresponds to *one* off-diagonal element represented by a membrane stretched between a fixed pair among the branes, and it is already obvious that diagonal and non-diagonal elements behave very differently (this can also be seen in an inherently 6-dimensional approach to constructing "non-abelian tensor multiplets" [21]). It would of course be nice to get more information concerning the relevance of topology in this and similar cases. If the above conjecture, that already a cylindrical membrane captures all boundary topologies, is true, it would be part of the answer. On the other hand, on should not *a priori* exclude states described by membranes ending on more than two branes, unless it can be argued that they are already in some way included.

The separation of the branes, the stretching, introduces a length scale which can be used to define a dimensionless coupling constant g as in section 4. It may be used for a perturbation expansion, where small coupling (large L) corresponds to strongly coupled string theory in the cases where such a correspondence exists (type IIA and heterotic).

The non-commutativity parameter, which in string theory has a physical meaning, acts in the present context as a device to deform the algebra to something that allows a consistent truncation. Whether or not it has a more direct physical interpretation is unclear. It is known that matrix theory, for finite and fixed N, breaks Lorentz invariance [22]. This fact can be traced directly to the manifestly non-covariant introduction of non-commutativity. To our knowledge it has never been shown, however, that Lorentz invariance has to be broken if θ is allowed to transform. It has been discussed that the non-commutativity should be lifted to some covariant "3-dimensional" structure on the membrane world-volume, see e.g. refs. [23, 24, 25]. Another question concerning the algebraic structure of the theory is if there is some physical situation that, e.g. via orientifolding, would lead to twisted affine algebras.

An interesting feature, not present in matrix theory, is that the symmetry algebra is large enough to allow for quantum mechanical anomalies. It would be very interesting if demanding that the symmetry remains unbroken by quantum effects could lead to a calculation of a critical dimension for the supermembrane.

Acknowledgments

This work is supported in part by the Swedish Science Council and by EU contract HPRN-CT-2000-00122. The author wants to thank Bengt E.W. Nilsson and Måns Henningson for discussions.

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